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Simultaneous Approximations for Functions in Sobolev Spaces by Derivatives of Polyharmonic Cardinal Splines*

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We prove in this paper that functions in Sobolev spaces and their derivatives can be approximated by polyharmonic splines and their derivatives in $L^p(\mathbb{R}^n)$ norms. Of particular interest are the remainder formulas of such approximations and the order of convergence by the derivatives of cardinal polyharmonic interpolational splines. © 1999 Academic Press

Key Words: polyharmonic spline; cardinal interpolation; remainder formula; approximation; Sobolev spaces; order of convergence; Peano type kernel.

1. INTRODUCTION

Approximations for functions in Sobolev spaces by other classes of functions are important in both function space theory and approximation theory. These areas have been extensively studied over the years, by both researchers in function space theory and experts in approximation theory.

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In particular, functions in Sobolev spaces in $\mathbb{R}^n$ have shown to be approximated nicely by polynomials, trigonometric polynomials, and splines. The main purpose of this paper is to approximate functions in Sobolev spaces by polyharmonic splines and some of their derivatives simultaneously.

Let Δ be the usual Laplace operator defined by

$$\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}, \tag{1.1}$$

and Δ^k, $k > 1$, denote its kth iterate, $\Delta^k u = \Delta(\Delta^{k-1} u)$. For $k \geqslant (n+1)/2$, we denote by $SH^k(\mathbb{R}^n)$ the subspace of $\mathfrak{S}'(\mathbb{R}^n)$ whose elements f enjoy the properties

(i) f is in $C^{2k-n-1}(\mathbb{R}^n)$,

(ii) $(\Delta^k f)(x) = 0$ for all $x \in \mathbb{R}^n \backslash \mathbb{Z}^n$,

where $\mathbb{Z}^n$ denotes the integer lattice in $\mathbb{R}^n$, and $\mathfrak{S}(\mathbb{R}^n)$ and $\mathfrak{S}'(\mathbb{R}^n)$ (see, e.g, [N, S]) denote respectively the Schwartz space of rapidly decreasing functions and its dual, the space of tempered distributions. According to Madych and Nelson [MN], a function or a distribution f is called a k-harmonic cardinal spline if $f \in SH^k(\mathbb{R}^n)$, and called a polyharmonic spline if $f \in SH^k(\mathbb{R}^n)$ for some k. Polyharmonic functions of infinite order were extensively studied in the monograph by Aronszajn *et al.* [ACL].

In [MN], Madych and Nelson gave the cardinal interpolations

$$(Sf)(x) = \sum_{v \in \mathbb{Z}^n} f(v)\, L_k(x - v), \tag{1.2}$$

where the cardinal function $L_k \in SH^k(\mathbb{R}^n)$ satisfies $L_k(v) = \delta_{0v}$ for all $v \in \mathbb{Z}^n$ and $\delta_{0v} = 0$ for $v \neq 0$ and 1 for $v = 0$.

This is a generalization of cardinal interpolation from the univariate spline theory of odd order formed by Schoenberg [Sc2] to the multivariate case. As a special case of multivariate cardinal interpolation with radial basis functions, Buhmann [B] considered the interpolations

$$(S_{\sigma,k} f)(x) = \sum_{v \in \mathbb{Z}^n} f\left(\frac{v}{\sigma}\right) L_k(\sigma x - v), \qquad \sigma > 0, \tag{1.3}$$

as an approximation to f. Exploiting the polynomial properties of the cardinal interpolation (1.2), Buhmann in [B] derived orders of convergence of (1.3) to a sufficiently differentiable f in the uniform norm when $\sigma \to +\infty$.

In [L1], the first author of the current article gave the remainder formula of the cardinal interpolation $S_{1,k} f$ as an approximation to f, which in the univariate case was found by Schoenberg [Sc1], and Micchelli [Mi]. By analyzing the remainder formula in multivariate case, in [L1] the orders of convergence of (1.3) were derived to a sufficiently differentiable f in the metric $L_p(\mathbb{R}^n)$ when $\sigma \to +\infty$. In [L2], Liu also gave some $L_2(R^n)$ extremal properties of cardinal natural polyharmonic interpolation.

Motivated by these works, several very interesting questions thus arise: Can we approximate simultaneously the derivatives of sufficiently smooth functions by the derivatives of the cardinal spline functions? Does there exist a remainder formula for such an approximation? What are the precise orders of such an approximation?

The main purpose of the present paper is to answer the above questions affirmatively to some extent. More precisely, we consider the approximation properties of certain particular derivatives of functions in $S_{1,k} f$, namely the powers of Laplacians. We use $L_p(\mathbb{R}^n)$ to denote the space of all p-power integrable functions f on $\mathbb{R}^n$ for $1 \leqslant p < +\infty$ and $L_\infty(\mathbb{R}^n)$ to denote the space of all essentially bounded functions f with the norms

$$\|f\|_p = \left(\int_{\mathbb{R}^n} |f(t)|^p \, dt \right)^{1/p}, \qquad 1 \leqslant p < \infty,$$

$$\|f\|_\infty = \operatorname*{esssup}_{t \in \mathbb{R}^n} |f(t)|,$$

respectively.

Before we state our main theorems, we need to introduce some notation. For $y = (t_1, ..., t_n) \in \mathbb{R}^n$ and $x = (x_1, ..., x_n) \in \mathbb{R}^n$, set $tx = \sum_{j=1}^n t_j x_j$. The Fourier transform $\hat{\varphi}$ of $\varphi \in \mathfrak{S}(\mathbb{R}^n)$ will be denoted as

$$(F\varphi)(x) = \hat{\varphi}(x) =: (2\pi)^{-n/2} \int_{\mathbb{R}^n} \varphi(t) \, e^{-itx} \, dt$$

and the transform inverse to it as

$$(F^{-1}\varphi)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \varphi(t) \, e^{itx} \, dt.$$

The generalized Fourier transforms (direct and inverse) for $f \in \mathfrak{S}'(\mathbb{R}^n)$ are defined respectively by the equations

$$(\hat{f}, \varphi) = (f, \hat{\varphi}) \qquad \text{and} \qquad (F^{-1}f, \varphi) = (f, F^{-1}\varphi) \tag{1.4}$$

for all $\varphi \in \mathfrak{S}(\mathbb{R}^n)$, where $(g, \varphi) = \int_{\mathbb{R}^n} g(t) \, \varphi(t) \, dt$, $g \in \mathfrak{S}'(\mathbb{R}^n)$, $\varphi \in \mathfrak{S}'(\mathbb{R}^n)$.

For $2k \geqslant n+1$, we denote the fundamental solution to k-Laplace by

$$E_k(x) = \begin{cases} c(n, k)\, \|x\|^{2k-n}, & \text{if } n \text{ is odd}, \\ c(n, k)\, \|x\|^{2k-n} \log \|x\|, & \text{if } n \text{ is even}, \end{cases} \tag{1.5}$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$, $c(n, k)$ is a constant which depends only on n and k, and is chosen such that $\Delta^k E_k(x) = \delta(x)$. Here $\Delta(x)$ denotes the unit Dirac distribution at the origin. The generalized Fourier transform of E_k is

$$\hat{E}_k(\xi) = (2\pi)^{-n/2}\, (-\|\xi\|^2)^{-k}. \tag{1.6}$$

We now sketch the main results of the paper and the main ideas of proving them. First of all, we will prove a remainder formula for functions $f \in L_p^{2k+2l}(\mathbb{R}^n)$, $2k \geqslant n+1$, as such:

$$(\Delta^l f)(x) - (\Delta^l S_{1,k+l} f)(x) = \int_{\mathbb{R}^n} G_{kl}(x, t)\, \Delta^{k+l} f(t)\, dt, \qquad \forall x \in \mathbb{R}^n.$$

Here the function $G_{kl}(x, t)$ is defined by the formula

$$G_{kl}(x, t) = E_k(x-t) - \sum_{v \in \mathbb{Z}^n} E_{k+l}(v-t)(\Delta^l L_{k+l})(x-v), \qquad (x, t) \in \mathbb{R}^n \times \mathbb{R}^n,$$

and satisfies some properties (see Theorem 1 in Section 2 for details). These properties of $G_{kl}(x, t)$ are derived by using Fourier transform estimates.

Second, we will derive the order of convergence of $\Delta^l f - \Delta^l S_{\sigma,k+l} f$ in $L^p(\mathbb{R}^n)$. More precisely we will prove that for $1 \leqslant p \leqslant \infty$ and $2k \geqslant n+1$, and $f \in L_p^{2k+2l}(\mathbb{R}^n)$,

$$\|\Delta^l f - \Delta^l S_{\sigma,k} f\|_p \leqslant \sigma^{-2k} C_{k,l,p} \|\Delta^{k+l} f\|_p,$$

holds, where the constant $C_{k,l,p}$ depends only on k, l and p (see Theorem 2 in Section 3 for details).

Finally, as a corollary of our main results we also obtain in Section 3

$$\sup_{f \in W_p^{k+l}(\Delta)} \|\Delta^l f - \Delta^l S_{\sigma,k+l} f\|_p \asymp \sigma^{-2k},$$

where the set $W_p^{k+l}(\Delta)$ is defined by

$$W_p^{k+l}(\Delta) = \{ f \in L_p^{2k+2l}(R^n) : \|\Delta^{k+l} f\|_p \leqslant 1 \},$$

and $A \asymp B$ denotes two quantities A and B are comparable in the sense that $c_1 A \leqslant B \leqslant c_2 B$ with two absolute constants.

2. REMAINDER FORMULA OF APPROXIMATION OF $\Delta^l f$ BY $\Delta^l S_{1,k+l} f$

In what follows we say that a sequence a_ν, $\nu \in \mathbb{Z}^n$, is of polynomial growth if there are positive constants c and N such that

$$|a_\nu| \leqslant c(1+\|\nu\|)^N \qquad \text{for all} \quad \nu \in \mathbb{Z}^n.$$

Similarly we say that a locally bounded function f decays exponentially if

$$|f(x)| \leqslant ce^{-N\|x\|} \qquad \text{for all} \quad x \in \mathbb{R}^n.$$

Madych and Nelson in [MN] obtained the following results.

THEOREM A. *If $v=\{v_j\}_{j\in\mathbb{Z}^n}$ is a sequence of polynomial growth, then there is a unique k-harmonic spline f of polynomial growth such that $f(j)=v_j$ for all $j\in\mathbb{Z}^n$. Every k-harmonic spline f has a unique representation in terms of translates of L_k, namely*

$$f(x)=\sum_{j\in\mathbb{Z}^n} f(j)\, L_k(x-j). \tag{2.1}$$

Here the function L_k is defined by its Fourier transform

$$\hat{L}_k(\xi)=(2\pi)^{-n/2}\frac{\|\xi\|^{-2k}}{\sum_{j\in\mathbb{Z}^n}\|\xi-2\pi j\|^{-2k}}, \qquad \xi\in\mathbb{R}^n, \tag{2.2}$$

satisfies the properties $L_k(\nu)=\delta_{0,\nu}$, for all $\nu\in\mathbb{Z}^n$, and decays exponentially.

For

$$\alpha=(\alpha_1, ..., \alpha_n)\in\mathbb{Z}^n_+ =: \{(\alpha_1, ..., \alpha_n) : \alpha_j=0, 1, 2, ..., j=1, 2, ..., n\},$$

let

$$\frac{\partial^\alpha}{\partial x^\alpha}=\frac{\partial^{\alpha_1+\alpha_2+\cdots+\alpha_n}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_n^{\alpha_n}}$$

be a differential monomial, whose total order is $|\alpha|=\alpha_1+\cdots+\alpha_n$. Suppose that f and g are two locally integrable functions on R^n. Then we say that the generalized derivative of f, $\partial^\alpha f/\partial x^\alpha$ is g, if

$$\int_{\mathbb{R}^n} f(x)\frac{\partial^\alpha\varphi(x)}{\partial x^\alpha}\,dx=(-1)^{|\alpha|}\int_{\mathbb{R}^n} g(x)\,\varphi(x)\,dx, \qquad \forall\varphi\in C_0^\infty(R^n),$$

where $C_0^\infty(\mathbb{R}^n)$ denotes the space of indefinitely differentiable functions with compact support (see [S]).

For $1 \leqslant p \leqslant \infty$, and a non-negative integer r, the Sobolev space $L_p^r(\mathbb{R}^n)$ is defined as the space of functions f, with $f \in L_p(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ and where all $\partial^\alpha f/\partial x^\alpha$ exist and $\partial^\alpha f/\partial x^\alpha \in L^p(\mathbb{R}^n)$ in the above sense, where $|\alpha| \leqslant r$.

In [L1], Liu obtained

THEOREM B. *If $f \in L_p^{2k}(\mathbb{R}^n)$, $2k \geqslant n+1$, then the remainder formula*

$$f(x) - S_{1,k} f(x) = \int_{\mathbb{R}^n} G_k(x,t)\, \Delta^k f(t)\, dt, \qquad \forall x \in \mathbb{R}^n,$$

holds. Here the function $G_k(x,t)$ is defined by the formula

$$G_k(x,t) = E_k(x-t) - \sum_{v \in \mathbb{Z}^n} E_k(v-t)\, L_k(x-v), \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}^n,$$

and satisfies the following properties:

(i) $|G_k(x,t)| \leqslant A(x)\, e^{-B\|t\|}$ *for all* $(x,t) \in \mathbb{R}^n \times \mathbb{R}^n$, *where B is a positive constant.*

(ii) $G_k(x-v,t) = G_k(x,t+v)$, *for all* $v \in \mathbb{R}^n$.

(iii) $G_k(v,t) = 0$, *for all* $v \in \mathbb{Z}^n$.

(iv) $G_k(x,t) = G_k(t,x)$ *for all x and t in* $\mathbb{R}^n$.

One of the main theorems of this section is

THEOREM 1. *If $f \in L_p^{2k+2l}(\mathbb{R}^n)$, $2k \geqslant n+1$, then the remainder formula*

$$(\Delta^l f)(x) - (\Delta^l S_{1,k+l} f)(x) = \int_{\mathbb{R}^n} G_{kl}(x,t)\, \Delta^{k+l} f(t)\, dt, \qquad \forall x \in \mathbb{R}^n, \tag{2.3}$$

holds. Here the function $G_{kl}(x,t)$ is defined by the formula

$$G_{kl}(x,t) = E_k(x-t) - \sum_{v \in \mathbb{Z}^n} E_{k+l}(v-t)(\Delta^l L_{k+l})(x-v), \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}^n, \tag{2.4}$$

and satisfies the following properties:

(i) $|G_{kl}(x,t)| \leqslant A(x)\, e^{-B\|t\|}$ *for all* $(x,t) \in \mathbb{R}^n \times \mathbb{R}^n$, *where B is a positive constant depending on k, l, n and $A(x)$ is a function of x (see definition in the proof of Lemma 2 in Section 2).*

(ii) $G_{kl}(x-v,t) = G_{kl}(x,t+v)$, *for all* $v \in \mathbb{R}^n$.

(iii) $G_{kl}(x,t) = G_{kl}(t,x)$ *for all x and t in* $\mathbb{R}^n$.

Remark 1. The special case in one-dimension when no derivatives were taken (i.e., $n=1$ and $l=0$) was considered by Micchelli [Mi] and Schoenberg [Sc2].

Remark 2. It is clear to see that Theorem B is a special case of Theorem 1 by taking $l=0$. The proof given below also gives a proof of Theorem B when $l=0$.

Remark 3. The referee points out to us that Theorems B and 1 are Peano kernel type theorems in a noncompact domain setting and infinite-dimensional space of functions. This actually describes the Peano kernel of a functional vanishing on polyharmonic functions of certain degrees. Peano type kernels for functionals vanishing on polyharmonic functions in compact domains were first considered in the paper by Haussmann and Kounchev [HK1, HK2]. We thank the referee for pointing out this to us.

Remark 4. It is not clear to us if the current proof of Theorem 1 shall work for arbitrary derivatives rather than the powers of the Laplacian Δ^l.

To prove Theorem 1, we need some lemmas as follows.
For all ξ and η in $\mathbb{R}^n$, put

$$z=\xi+i\eta=(\xi_1+i\eta_1, ..., \xi_n+i\eta_n)=(z_1, ..., z_n),\ q(z)=-\sum_{m=1}^{n} z_m^2,$$

$$E_j(x, z)=\sum_{v\in\mathbb{Z}^n}\frac{e^{2\pi i v x}}{\{q(z+2v\pi)\}^j}, \qquad (2j>n),$$

$$K_{kl}(x, z)=\frac{e^{ixz}\{[q(z)]^l E_{k+l}(0, z)-E_k(x, z)\}}{[q(z)]^{k+l} E_{k+l}(0, z)}, \tag{2.5}$$

for all $x\in\mathbb{R}^n$, in which when $z=2\pi j$, $K_{kl}(x, 2\pi j)$ is defined by $K_{kl}(x, 2\pi j)=: \lim_{z\to 2\pi j} K_{kl}(x, z)$.

LEMMA 1. *Let $K_{kl}(x, z)$ be defined as above. Then*

(i) $$K_{kl}(x, \xi)=\frac{e^{ix\xi}(-\|\xi\|^2)^l-\sum_{v\in\mathbb{Z}^n} e^{iv\xi}(\Delta^l L_{k+l})(x-v)}{(-\|\xi\|^2)^{k+l}},$$

for all $(x, \xi)\in\mathbb{R}^n\times\mathbb{R}^n$;

(ii) *there exists some $\varepsilon>0$ such that for each x fixed, as a function of z, $K_{kl}(x, z)$ is analytic in the tube*

$$\mathbb{R}^n_\varepsilon =: \{z=\xi+i\eta : \xi\in\mathbb{R}^n,\ -\varepsilon<\eta_m<\varepsilon,\ m=1, 2, ..., n\};$$

(iii) *for each* $z \in \mathbb{R}^n_\varepsilon$, $K_{kl}(x, z)$ *as a function of* x *belongs to* $C^{2k-n-1}(\mathbb{R}^n)$.

Proof. First, we prove that (i) is true. For each $\xi \in \mathbb{R}^n$, put

$$g(x) = e^{-ix\xi} \sum_{v \in \mathbb{Z}_n} e^{iv\xi} (\Delta^l L_{k+l})(x - v). \tag{2.6}$$

Since $\sum_{v \in Z^n} L_{k+l}(x-v) \equiv 1$, it is easy to see that when $\xi = 2m\pi$, $m \in \mathbb{Z}^n$, $g(x) \equiv 0$ for the integral number $l \geqslant 1$, and when $\xi \in \mathbb{R}^n - 2\pi\mathbb{Z}^n$, $g(x)$ is a periodic function; i.e., $g(v+x) = g(x)$ for all $v \in \mathbb{Z}^n$. Then, we have the Fourier series expansion of g as follows:

$$g(x) = \sum_{m \in \mathbb{Z}^n} \left(\frac{(-\|\xi + 2\pi m\|^2)^{-k}}{E_{k+l}(0, \xi)} \right) e^{2\pi i m x}. \tag{2.7}$$

In fact, taking (2.2) into account, we have the Fourier coefficients of g,

$$\begin{aligned} \hat{g}(m) &=: \int_{[0,1]^n} g(x)\, e^{-2i\pi m x}\, dx \\ &= (2\pi)^{n/2} (-\|\xi + 2\pi m\|^2)^l\, \hat{L}_{k+l}(\xi + 2\pi m), \qquad \forall m \in \mathbb{Z}^n. \end{aligned}$$

By (2.7), we have

$$\begin{aligned} &\frac{e^{ix\xi}(-\|\xi\|^2)^l - \sum_{v \in \mathbb{Z}^n} e^{iv\xi} (\Delta^l L_{k+l})(x - v)}{(-\|\xi\|^2)^{k+l}} \\ &\quad = \frac{e^{ix\xi}(-\|\xi\|^2)^l - g(x)\, e^{ix\xi}}{(-\|\xi\|^2)^{k+l}} \\ &\quad = \frac{e^{ixz}\{[q(z)]^l E_{k+l}(0, z) - E_k(x, z)\}}{[q(z)]^{k+l} E_{k+l}(0, z)} = K_{kl}(x, z), \end{aligned}$$

which is (i).

In order to show the assertion (ii), we let $Q^n = (-\pi, \pi]^n$. The function

$$F(z) = \sum_{v \in \mathbb{Z}^n - \{0\}} \frac{1}{[q(z + 2v\pi)]^{k+l}} \tag{2.8}$$

is defined by an absolutely convergent series which is analytic on $Q^n \times i(-\varepsilon, \varepsilon)^n$ for some $\varepsilon > 0$, and the function

$$R(z) =: 1 + [q(z)]^{k+l} F(z) \tag{2.9}$$

has no zero on $Q^n \times i(-\varepsilon, \varepsilon)^n$. Thus, $K_{kl}(x, z)$ as a function of z is analytic on $Q^n \times i(-\varepsilon, \varepsilon)^n$ for any $x \in \mathbb{R}^n$ (see [MN] for similar arguments).

For $z \in (Q^n + 2\pi v) \times i(-\varepsilon, \varepsilon)^n$, $v \neq 0$, the function $K_{kl}(x, z)$ has the representation

$$K_{kl}(x, z) = \frac{e^{ixz}}{(q(z))^{k+l}} \left(\frac{[q(z)]^l - [q(z)]^l e^{2\pi i v x}}{1 + H_1(z)} + \frac{(q(z))^l H_1(z) - (q(z + 2\pi v))^l H_2(z)}{1 + H_1(z)} \right). \tag{2.10}$$

Here we have put

$$H_1(z) =: \sum_{j \in Z^n - \{v\}} \left(\frac{q(z + 2\pi v)}{q(z + 2\pi j)} \right)^{k+l}$$

$$H_2(z) =: \sum_{j \in Z^n - \{v\}} \left(\frac{q(z + 2\pi v)}{q(z + 2\pi j)} \right)^{k} e^{2\pi i v x}$$

By a similar argument as above, we may see that $K_{kl}(x, z)$ as a function of z is also analytic on the set $(Q^n + 2\pi v) \times i(-\varepsilon, \varepsilon)^n$, $v \neq 0$. Therefore, the assertion (ii) holds.

The properties of $L_{k+l}(x)$ imply that the assertion (iii) is obvious.

LEMMA 2. *For* $2k \geqslant n + 1$, *let*

$$G_{kl}(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} K_{kl}(x, u) \, e^{-itu} \, du. \tag{2.11}$$

Then, we have

(i) $|G_{kl}(x, t)| \leqslant A(x) \, e^{-B \|t\|}$, *for some* $B > 0$,

(ii) $G_{kl}(x, t) = E_k(x - t) - \sum_{v \in \mathbb{Z}^n} E_{k+l}(v - t)(\Delta^l L_{k+l})(x - v)$,

(iii) $G_{kl}(x, t) = G_{kl}(t, x)$, *for all* $x \in \mathbb{R}^n$, $t \in \mathbb{R}^n$.

(iv) $G_{kl}(x - v, t - v) = G_{kl}(x, t)$, *for all* $x \in [0, 1]^n$, $t \in [0, 1]^n$, $v \in Z^n$.

Proof. (i) For each $t \in \mathbb{R}^n$, we choose the signs of the components of $\gamma \in \mathbb{R}^n$ to satisfy $t\gamma = -\varepsilon/2 \sum_{j=1}^n |t_j|$. Then, by virtue of Lemma 1(ii), we can apply Cauchy's theorem (see [K]) in order to obtain

$$(2\pi)^n \, |G_{kl}(x, t)| = \left| \int_{\mathbb{R}^n} K_{kl}(x, u) \, e^{-itu} \, du \right|$$

$$= \left| \int_{\mathbb{R}^n + i\gamma} K_{kl}(x, z) \, e^{-itz} \, dz \right|$$

$$\leqslant e^{t\gamma} \int_{\mathbb{R}^n} |K_{kl}(x, \xi + i\gamma)| \, d\xi$$

$$\leqslant A(x) \, e^{-B \, \|t\|}, \tag{2.12}$$

which is the assertion (i). Here $B = \varepsilon/2 \sqrt{n}$, and

$$A(x) =: \sup \left\{ \int_{\mathbb{R}^n} |K_{kl}(x, \xi + i\gamma)| \, d\xi : |\gamma_j| \leqslant \frac{\varepsilon}{2}, \, j = 1, 2, ..., n \right\}.$$

By Lemma 1(i), (ii) of Lemma 2 is then obvious.

As to the assertion (iii), by the definition (2.11), we have

$$\begin{aligned} G_{kl}(x, t) &= \int_{\mathbb{R}^n} \frac{e^{ixy}((-\|y\|^2)^l \, E_{k+l}(0, y) - E_k(x, y))}{(-\|y\|^2)^{k+l} \, E_{k+l}(0, y)} \, e^{-iyt} \, dy \\ &= \sum_{m \in \mathbb{Z}^n} \int_{Q^n + 2m\pi} \frac{((-\|y\|^2)^l \, E_{k+l}(0, y) - E_k(x, y))}{(-\|y\|^2)^{k+l} \, E_{k+l}(0, y)} \, e^{i(x-t)y} \, dy \\ &= \int_{Q^n} \frac{E_k(x-t, u) \, E_{k+l}(0, u) - E_{k+l}(-t, u) \, E_k(x, u)}{E_{k+l}(0, u)} \, e^{i(x-t)u} \, du. \end{aligned} \tag{2.13}$$

Using the facts that $E_j(-t, u) = E_j(t, -u)$ and $E_j(0, u) = E_j(0, -u)$, for $j = k, k+l$, we see that $G_{kl}(x, t)$ has the property $G_{kl}(x, t) = G_{kl}(t, x)$, which is the assertion (iii). By (iii), (iv) is trivial

Lemma 3. *Let $f \in L_1^{2k+2l}(\mathbb{R}^n)$, $2k \geqslant n+1$. Then $\hat{f} \in L_1(\mathbb{R}^n)$.*

Proof of Theorem 1. When $p = 1$, using Lemma 1(i) and the fact that $G_{kl}(x, t) = [K_{kl}(x, \cdot)]^\wedge(t)(2\pi)^{-n/2}$, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^n} G_{kl}(x, t) \, \Delta^{k+l} f(t) \, dt \\ &\quad = (2\pi)^{-n/2} \int_{\mathbb{R}^n} K_{kl}(x, t) (\Delta^{k+l} f)^\wedge(t) \, dt \\ &\quad = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \left\{ (-\|u\|^2)^l \, e^{ixu} - \sum_{v \in \mathbb{Z}^n} e^{ivu} (\Delta^l L_{k+l})(x - v) \right\} \hat{f}(u) \, du \\ &\quad = (\Delta^l f)(x) - \sum_{v \in \mathbb{Z}^n} f(v) (\Delta^l L_{k+l})(x - v). \end{aligned} \tag{2.14}$$

When $1 < p \leqslant \infty$, let $g(x)$ in $C^\infty(\mathbb{R}^n)$ satisfy the following conditions:

$$\begin{cases} g(x) = 0 & \text{if} \quad \|x\| \geqslant 2, \\ g(x) \equiv 1 & \text{if} \quad \|x\| \leqslant 1, \\ |g(x)| \leqslant 1 & \forall x \in \mathbb{R}^n. \end{cases}$$

Let $f \in L_p^{2k+2l}(\mathbb{R}^n)$. For any $N \geqslant 1$, put

$$f_N(x) = g\left(\frac{x}{N}\right) f(x).$$

Then, we see that $f_N \in L_p^{2k+2l}(\mathbb{R}^n) \cap L_1^{2k+2l}(\mathbb{R}^n)$, $\Delta^l f_N(x) \to \Delta^l f_N(x)$ as $N \to \infty$, and

$$(\Delta^{k+l} f_N)(x) = (\Delta^{k+l} f)(x)\, g\left(\frac{x}{N}\right) + G_N(x).$$

Here the functions $G_N(x)$ satisfy properties $G_N(x) = 0$ if $|x| < N$ or $|x| > 2N$, and $\|G_N\|_{L_p(\mathbb{R}^n)} \to 0$ $(N \to \infty)$.

Using the fact that we have proved for $p = 1$, it is easy to verify that

$$\begin{aligned}
&(\Delta^l f)(x) - \sum_{v \in \mathbb{Z}^n} f(v)(\Delta^l L_{k+l})(x - v) \\
&\quad = \lim_{N \to \infty} \left(\Delta^l f_N(x) - \sum_{v \in \mathbb{Z}^n} f_N(v)\, \Delta^l L_{k+l}(x - v) \right) \\
&\quad = \lim_{N \to \infty} \int_{\mathbb{R}^n} G_{kl}(x, t) \left\{ (\Delta^{k+l} f)(t)\, g\left(\frac{t}{N}\right) + G_N(t) \right\} dt \\
&\quad = \int_{\mathbb{R}^n} G_{kl}(x, t)(\Delta^{k+l} f)(t)\, dt
\end{aligned} \tag{2.15}$$

Thus, we have completed the proof of Theorem 1.

3. ORDER OF CONVERGENCE OF $\Delta^l S_{\sigma, k+l} f$ TO $\Delta^l f$

In this section, we prove another main theorem of this paper, namely on the orders of convergence of approximation by polyharmonic splines and their derivatives.

THEOREM 2. *For* $1 \leqslant p \leqslant \infty$ *and* $2k \geqslant n+1$, *let* $f \in L_p^{2k+2l}(\mathbb{R}^n)$. *Then*

$$\|\Delta^l f - \Delta^l S_{\sigma,k+l} f\|_p \leqslant \sigma^{-2k} C_{k,l,p} \|\Delta^{k+l} f\|_p. \tag{2.16}$$

Here the constant $C_{k,l,p}$ *depends only on* k, l *and* p.

Proof. For each $\sigma > 0$, by Theorem 1, it is easy to see that

$$(\Delta^l f)(x) - (\Delta^l S_{\sigma,k+l} f)(x) = \frac{1}{\sigma^{2k-n}} \int_{\mathbb{R}^n} G_{kl}(\sigma x, \sigma t)(\Delta^{k+l} f)(t)\, dt \tag{2.17}$$

Next, we give the proof of Theorem 2 in the case $1 < p < \infty$. The cases for $p = 1, \infty$ are similar.

By Hölder's inequality, we have

$$\left| \int_{\mathbb{R}^n} G_{kl}(\sigma x, \sigma t)(\Delta^{k+l} f)(t)\, dt \right|$$

$$\leqslant \left\{ \int_{\mathbb{R}^n} |G_{kl}(\sigma x, \sigma t)|\, dt \right\}^{1/p'} \left\{ \int_{\mathbb{R}^n} |G_{kl}(\sigma x, \sigma t)|\, |\Delta^{k+l} f(t)|^p\, dt \right\}^{1/p}, \tag{2.18}$$

for $1 < p < \infty$, $1/p + 1/p' = 1$.

Hence, by (2.17) and (2.18), we obtain

$$\sigma^{(2k-n)p} \|\Delta^l f - \Delta^l S_{\sigma,k} f\|_p^p$$

$$\leqslant \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} |G_{kl}(\sigma x, \sigma t)|\, |\Delta^{k+l} f(t)|^p\, dt \right\} \left\{ \int_{\mathbb{R}^n} |G_{k+l}(\sigma x, \sigma t)|\, dt \right\}^{p-1} dx$$

$$= \sigma^{-np} \int_{\mathbb{R}^n} |\Delta^{k+l} f(t)|^p\, g(\sigma t)\, dt. \tag{2.19}$$

Here we have put

$$g(t) = \int_{\mathbb{R}^n} |G_{k+l}(x, t)| \left(\int_{\mathbb{R}^n} |G_{kl}(\tau, x)|\, d\tau \right)^{p-1} dx. \tag{2.20}$$

Notice that $g \in C(\mathbb{R}^n)$ and $g(t+v) = g(t)$ for all $v \in \mathbb{Z}^n$, we see that $g \in L_\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$. Hence, by (2.19), we have

$$\|\Delta^l f - \Delta^l S_{\sigma,k+l} f\|_p \leqslant \sigma^{-2k} C_{k,l,p} \|\Delta^{k+l} f\|_p, \tag{2.21}$$

which is (2.16) for $C_{k,l,p} =: \max_{x \in \mathbb{R}^n} |g(x)|$.

COROLLARY 3. *Let* $1 \leqslant p \leqslant \infty$ *and* $2k \geqslant n+1$. *Then,*

$$\sup_{f \in W_p^{k+l}(\Delta)} \|\Delta^l f - \Delta^l S_{\sigma, k+l} f\|_p \asymp \sigma^{-2k}.$$

Here the set $W_p^{k+l}(\Delta)$ *is defined by*

$$W_p^{k+l}(\Delta) = \{ f \in L_p^{2k+2l}(R^n) : \|\Delta^{k+l} f\|_p \leqslant 1 \}.$$

Proof. First, by Theorem 2, we have

$$\sup_{f \in W_p^{k+l}(\Delta)} \|\Delta^l f - \Delta^l S_{\sigma, k+l} f\|_p \ll \sigma^{-2k}.$$

On the other hand, by the results in [L1], we have

$$\| \sup_{f \in W_p^{k+l}(\Delta)} \|\Delta^l f - \Delta^l S_{\sigma, k+l} f\|_p$$

$$\geqslant \sup_{f \in W_p^{k+l}(\Delta)} \inf_{g \in SH_\sigma^k(R^n)} \|\Delta^l f - g\|_p \gg \sigma^{-2k}.$$

Here the set $SH_\sigma^k(R^n)$ is defined by

$$SH_\sigma^k(R^n) = \{ S \in C^{2k-n-1}(R^n) : (\Delta^k S)(x) = 0, \text{ for all } x \in R^n - \sigma^{-1} Z^n \}.$$

Thus, we have finished the proof of Corollary 3.

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